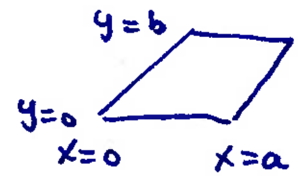
Laplace's Equation

Heat eq: $u_t = k u_{xx}$ (1-D)

$u_t = k(u_{xx} + u_{yy})$ (2-D)



Wave eq: $u_{tt} = a^2 u_{xx}$ (1-D)

$u_{tt} = a^2(u_{xx} + u_{yy})$ (2-D)

the right side: $u_{xx} + u_{yy}$, u_{xx}, etc (2nd partials w/ space variables)

all can be represented by the Laplacian operator ∇^2

if $u(x,y)$, then $\nabla^2 u = u_{xx} + u_{yy}$

$u(x,y,z)$, then $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

$\nabla^2 u$ tells us the shape of u and its relationship to the average nearby

$\nabla^2 u$ if $u = u(x)$ is u_{xx}

let's look at 2-D heat eq: $u_t = k \nabla^2 u = k(u_{xx} + u_{yy})$

steady-state $\to$ $u_t = 0$

$\to$ $\boxed{u_{xx} + u_{yy} = 0}$ <u>Laplace's</u> Eq.

set up: $0 < x < a$, $0 < y < b$

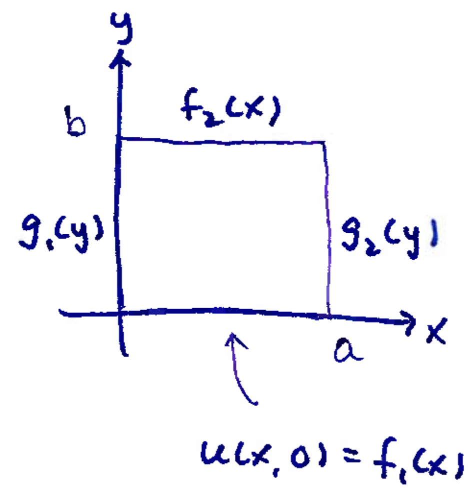
4 BCs: for each edge

$u(x,0) = f_1(x)$ (bottom)

$u(x,b) = f_2(x)$ (top)

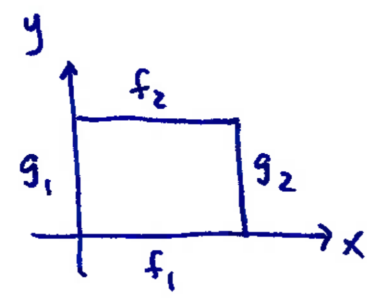
$u(0,y) = g_1(y)$ (left)

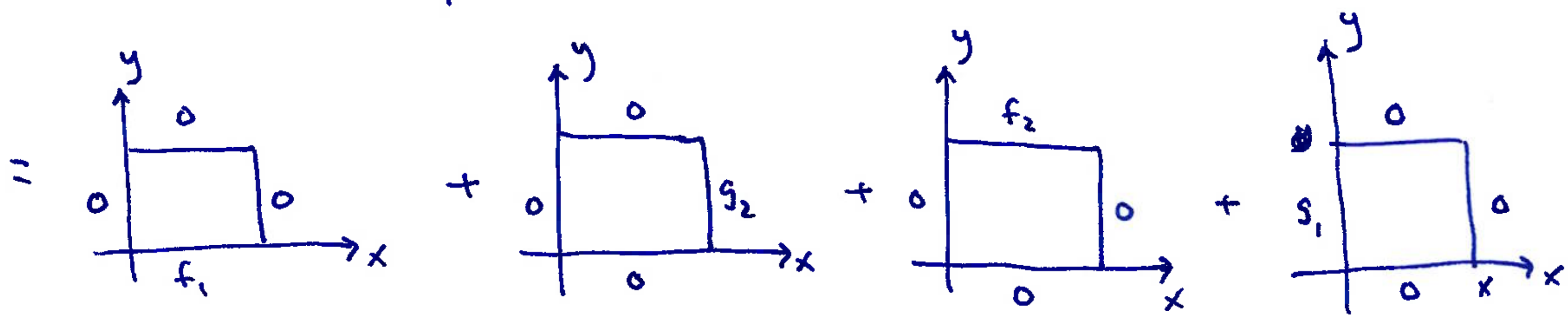
$u(a,y) = g_2(y)$ (right)



goal: find $u(x,y)$ that satisfies $u_{xx} + u_{yy} = 0$ AND all BCs the equation is <u>linear</u> so <u>superposition</u> applies

idea:





→ make 3 BCs homogeneous (zero), rotate ~~and~~, solve, combine.

as an example, let's solve the 3rd case above

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < a \qquad 0 < y < b$$

$u(x,0) = 0$ (bottom)

$u(0,y) = 0$ (left)

$u(a,y) = 0$ (right)

can be the insulated version w/ partial = 0

$u(x,b) = f(x)$ (top)

we will use separation of variables again

$$u(x,y) = X(x)Y(y) \qquad u_{xx} + \cancel{y}u_{yy} = 0$$

$$u_{xx} = X''Y \qquad u_{yy} = XY''$$

$$X''Y + XY'' = 0$$

$$X''Y = -XY'' \rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} = -\lambda$$

(same as in heat/wave)

two ODEs result:

$$X'' + \lambda X = 0$$

$$Y'' - \lambda Y = 0$$

BCs:

$$u(x,0) = 0 \rightarrow Y(0) = 0$$

$$u(0,y) = 0 \rightarrow X(0) = 0$$

$$u(a,y) = 0 \rightarrow X(a) = 0$$

Solve for X or Y first

whichever has complete BCs

(NOT always X first)

here, X first

$$X'' + \lambda X = 0 \qquad X(0) = X(a) = 0$$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{a^2}}$$

a is L here

$$\boxed{X_n = \sin\left(\frac{n\pi x}{a}\right)} \qquad n = 1, 2, 3, \ldots$$

now $\quad Y'' - \lambda Y = 0 \qquad Y(0) = 0$

$$Y'' - \frac{n^2\pi^2}{a^2} Y = 0$$

$$Y(y) = A e^{\frac{n\pi}{a} y} + B e^{-\frac{n\pi}{a} y}$$

or

$$Y(y) = c_1 \cosh\left(\frac{n\pi}{a} y\right) + c_2 \sinh\left(\frac{n\pi}{a} y\right)$$

choose the form that is convenient for the BC

~~y(0)=~~

$$Y(0) = c_1 \rightarrow \boxed{Y_n = \sinh\left(\frac{n\pi}{a} y\right)}$$

for each n, $u_n = X_n Y_n$

general solution: $\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)}$

last BC: $u(x,b) = f(x)$ (top)

$$f(x) = \sum_{n=1}^{\infty} \left[A_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right) \quad \text{sine series}$$

$$A_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

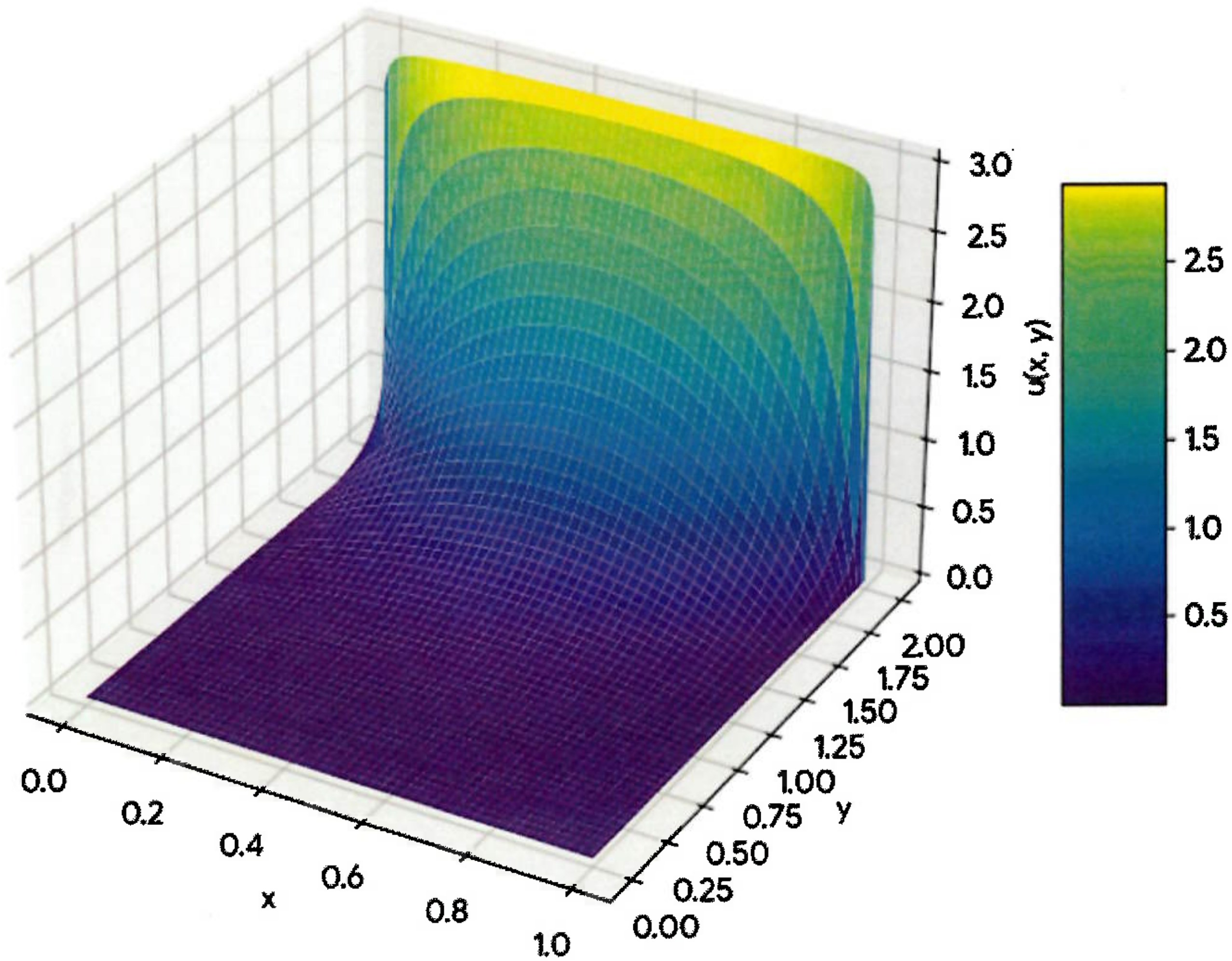
"L"
for x

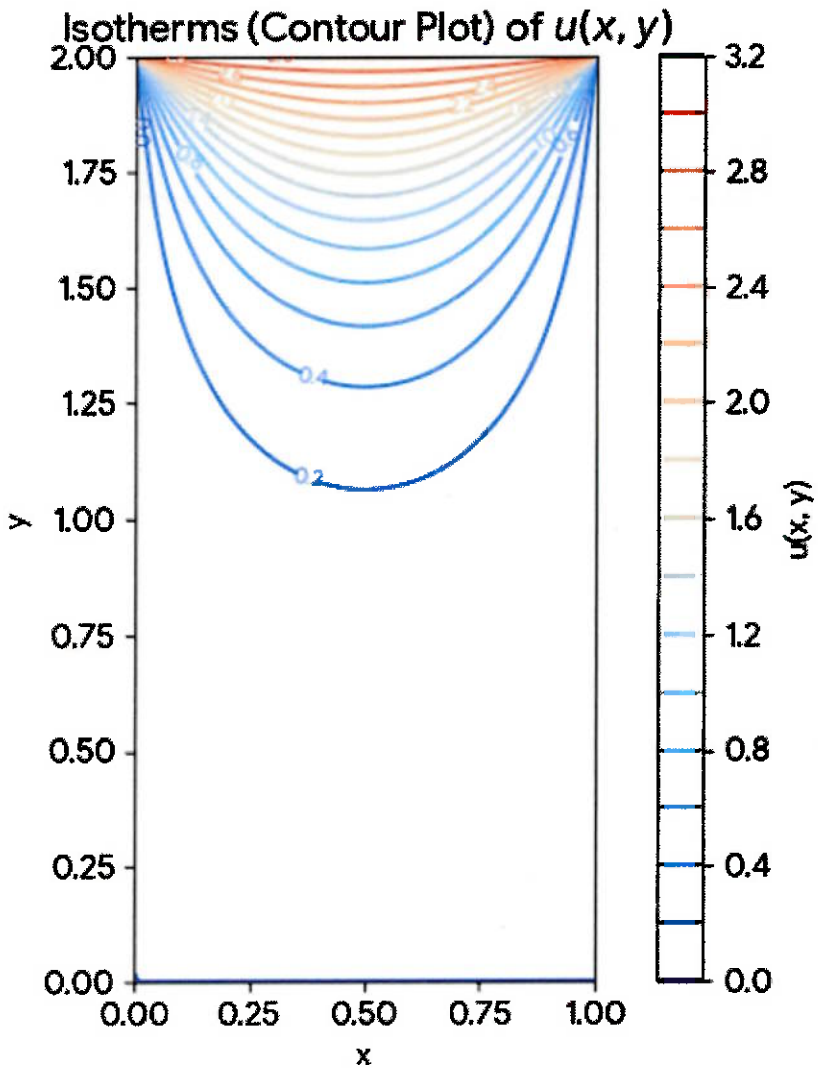
A_n comes from that: $\boxed{A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx}$

$a=1, \ b=2$

$f(x)=3$ (top edge)

Surface Plot of $u(x, y)$ (Laplace Equation Solution)





Isotherms (Contour Plot) of $u(x, y)$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.00
0.25
0.50
0.75
1.00
x
0.4
0.2
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0.4
0.0
u(x, y)